

Distance signless Laplacian spectral radius and Hamiltonian properties of graphs *

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Abstract

In this paper, first, we establish a sufficient condition for a bipartite graph to be Hamilton-connected. Furthermore, we also give two sufficient conditions on distance signless Laplacian spectral radius for a graph to be Hamilton-connected and traceable from every vertex, respectively. Last, we obtain a sufficient condition for a graph to be Hamiltonian in terms of the distance signless Laplacian spectral radius of G^C .

Key Words: Hamilton-connected, Traceable from every vertex, Distance signless Laplacian spectral radius.

AMS Subject Classification (1991): 05C50, 15A18, 05C38.

1 Introduction

In this paper, we only consider finite simple undirected graphs. We denote by m the edge number of G , δ the minimum degree of G , $d_G(v)$ or simply $d(v)$ the degree of v in G . We use $G[X, Y]$ to denote a bipartite graph with bipartition (X, Y) . Let K_n be a complete graph of order n and $K_{m,n}$ be a complete bipartite graph with two parts having m, n vertices. For two disjoint graphs G_1 and G_2 , the union of G_1 and G_2 , denoted by $G_1 + G_2$, is defined as $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, $E(G_1 + G_2) = E(G_1) \cup E(G_2)$; and the join of G_1 and G_2 , denoted by $G_1 \vee G_2$, is defined as $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$, and $E(G_1 \vee G_2) = E(G_1 \vee G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. The union of k disjoint copies of the same graph G is denoted by kG . Write $K_{n-1} + e$ for the complete graph on $n - 1$ vertices with a pendant edge, and $K_{n-1} + v$ for the complete graph on $n - 1$ vertices together with an isolated vertex. The complement G^C of G is the graph on $V(G)$ with edge set $[V]^2 \setminus E(G)$.

The distance between u and v in G , denoted by $d_G(u, v)$, is the length of a shortest path from u to v . The transmission $Tr(u)$ of a vertex u is defined to be the sum of distances from u to all other vertices in G , i.e., $Tr(u) = \sum_{v \in V(G)} d_G(u, v)$. A graph G is said to be transmission regular if $Tr(u)$ is a constant for each $u \in V(G)$. The transmission of G is the sum of distances between every pair of vertices of G . We denote it by $\sigma(G)$. Obviously, we

*Supported by the National Natural Science Foundation of China (No.11171273)

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have $\sigma(G) = \frac{1}{2} \sum_{u \in V(G)} Tr(u)$. The distance matrix of G , denoted by $\mathcal{D}(G)$, is a symmetric real matrix with (i, j) -entry being $d(v_i, v_j)$. It is easy to see that $Tr(v_i)$ is the sum of i -th row of $\mathcal{D}(G)$. Let $Tr(G)$ be the diagonal matrix of the vertex transmissions in G . Distance signless Laplacian matrix of graph G is defined as $Q_D(G) = Tr(G) + \mathcal{D}(G)$. The largest eigenvalue of $Q_D(G)$, denoted by $\rho_D(G)$, is called to be the distance signless Laplacian spectral radius of G .

Hamilton cycle (path) is a cycle (path) that passes through all the vertices of a graph. A graph is Hamiltonian (traceable) if it contains a Hamilton cycle (Hamilton path). And a graph is Hamilton-connected if every two vertices of G are connected by a Hamilton path. A graph is traceable from a vertex x if it has a Hamilton x -path.

Determining whether a given graph is Hamiltonian or not is an old problem in graph theory. This problem is proved to be an NP-hard problem [7]. Many graph theorists are interested in finding sufficient conditions for Hamilton cycles in graphs for a long time. In recent years, graph theorists tried to use spectral graph theory to solve this problem. Of course, there are many sufficient conditions on spectral radius or the signless Laplacian spectral radius for a graph to be Hamiltonian, traceable or Hamilton-connected. In 2003, Krivelevich and Sudakov first proposed a sufficient condition on the spectrum of the adjacency matrix for a regular graph to be Hamiltonian, where the graphs satisfying the given condition are pseudo-random. Some other spectral conditions for Hamilton cycles and paths in graphs have been given in [1, 2, 4, 5, 6, 8, 11, 9, 13, 14]. In this paper, we mainly consider the relationship between distance signless Laplacian spectral radius and the Hamiltonian properties of graphs. In other words, we try to use distance signless Laplacian spectral radius to judge whether a graph is Hamilton-connected or not. And we also give three sufficient conditions in terms of distance signless Laplacian spectral radius of G and one sufficient condition in terms of distance signless Laplacian spectral radius of G^C .

2 Lemmas and Results

Let $H_{t,n-t}$ ($t \geq 1$) be a bipartite graph obtained from $K_{n,n-t}$ by adding t vertices which are adjacent to t common vertices with degree $n-t$ in $K_{n,n-t}$, respectively. As for Lemma 2.1, Moon and Moser in [10] obtained the strict inequality. Ferrara, Jacobson and Powell in [3] characterized maximal nonhamiltonian bipartite graphs.

Lemma 2.1. ([10, 3]). *Let $G = G[X, Y]$ be a bipartite graph with minimum degree $\delta \geq t$ ($t \geq 1$) and m edges, where $|X| = |Y| = n \geq 2t$. If*

$$m \geq n^2 - tn + t^2,$$

then G is Hamiltonian unless $G = H_{t,n-t}$.

Lemma 2.2. ([12]). *Let $G = G[X, Y]$ be a bipartite graph on n vertices, then*

$$\rho_D(G) \geq \begin{cases} 3n - 4, & \text{if } n \text{ is even} \\ \frac{5n-8+\sqrt{n^2+8}}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Theorem 2.3. *Let $G[X, Y]$ be a bipartite graph with minimum degree $\delta \geq t$ ($t \geq 1$) and m edges, where $|X| = |Y| = n \geq 2t$. If*

$$\rho_D(G) \leq m - n^2 + (t + 6)n - (t^2 + 4),$$

then G is Hamiltonian unless $G = H_{t, n-t}$.

Proof. Because $|X| = |Y| = n \geq 2t$, G is a bipartite graph with order $2n$. By Lemma 2.2, we have $\rho_D(G) \geq 6n - 4$. Combining with the conditions of Theorem 2.3, we get

$$6n - 4 \leq \rho_D(G) \leq m - n^2 + (t + 6)n - (t^2 + 4),$$

then $m \geq n^2 - tn + t^2$. By Lemma 2.1, we obtain that G is Hamiltonian unless $G = H_{t, n-t}$.

The proof is complete. \blacksquare

Lemma 2.4. ([12]). *Let G be a connected graph on n vertices, then*

$$\rho_D(G) \geq \frac{4\sigma(G)}{n},$$

with equality holds if and only if G is transmission regular.

Let $\mathbb{NP}_1 = \{K_3 \vee (K_{n-5} + 2K_1), K_6 \vee 6K_1, K_4 \vee (K_2 + 3K_1), 5K_1 \vee K_5, K_4 \vee (K_{1,4} + K_1), K_4 \vee (K_{1,3} + K_2), K_3 \vee K_{2,5}, K_4 \vee 4K_1, K_3 \vee (K_1 + K_{1,3}), K_3 \vee (K_{1,2} + K_2), K_2 \vee K_{2,4}\}$.

Lemma 2.5. ([15]). *Let G be a connected graph on $n \geq 5$ vertices and m edges with minimum degree $\delta \geq 3$. If*

$$m \geq \binom{n-2}{2} + 6,$$

then G is Hamilton-connected unless $G \in \mathbb{NP}_1$.

G is a graph of order n , if X is an eigenvector of $Q_D(G)$ corresponding to eigenvalue ρ , then there is a 1-1 map φ from $V(G)$ to the entries of X , simply written as $X_u = \varphi(u)$ for each $u \in V(G)$. X_u is also called the value of u given by X . We can find that

$$[\rho - \text{Tr}(u)]X_u = \sum_{v \in V(G)} d_G(u, v)X_v, \quad (1)$$

for each $u \in V(G)$.

Theorem 2.6. *Let G be a connected graph on $n \geq 5$ vertices with minimum degree $\delta \geq 3$. If*

$$\rho_D(G) \leq \frac{2n^2 + 6n - 36}{n},$$

then G is Hamilton-connected.

Proof. Let $v \in V(G)$, $d(v)$ be the degree of v in G . Then

$$\text{Tr}(v) \geq d(v) \cdot 1 + (n - 1 - d(v)) \cdot 2 = 2(n - 1) - d(v),$$

with equality holds if and only if the maximum distance between v and other vertices in G is at most 2. So,

$$\sigma(G) = \frac{1}{2} \sum_{v \in V(G)} \text{Tr}(v) \geq \frac{1}{2} \sum_{v \in V(G)} [2(n - 1) - d(v)] = n(n - 1) - m,$$

Table 1: Some data of some graphs in \mathbb{NP}_1 .

G	$\rho_D(G)$	n	$\frac{2n^2+6n-36}{n}$
$K_6 \vee 6K_1$	28.8102	12	27
$K_4 \vee (K_2 + 3K_1)$	21.2319	9	20
$5K_1 \vee K_5$	23.4031	10	22.4
$K_4 \vee (K_{1,4} + K_1)$	23.8062	10	22.4
$K_4 \vee (K_{1,3} + K_2)$	23.5751	10	22.4
$K_3 \vee K_{2,5}$	23.5751	10	22.4
$K_4 \vee 4K_1$	18	8	17.5
$K_3 \vee (K_{1,3} + K_1)$	18.5208	8	17.5
$K_3 \vee (K_{1,2} + K_2)$	18.2789	8	17.5
$K_2 \vee K_{2,4}$	18.2381	8	17.5

with equality holds if and only if the maximum distance between v and other vertices in G is at most 2. By Lemma 2.4, we get

$$\rho_D(G) \geq \frac{4\sigma(G)}{n} \geq 4(n-1) - \frac{4m}{n}.$$

Then by the conditions of Theorem 2.6, we have

$$4(n-1) - \frac{4m}{n} \leq \rho_D(G) \leq \frac{2n^2 + 6n - 36}{n},$$

then $m \geq \binom{n-2}{2} + 6$. Suppose that G is not Hamilton-connected, by Lemma 2.5, we have $G \in \mathbb{NP}_1$. By direct calculation (see Table 1), we obtain that all graphs in \mathbb{NP}_1 except $G = K_3 \vee (K_{n-5} + 2K_1)$ satisfy $\rho_D(G) \geq \frac{2n^2+6n-36}{n}$, so we can get a contradiction.

As for $G = K_3 \vee (K_{n-5} + 2K_1)$, let $X = (x_1, x_2, \dots, x_n)^T$ be the eigenvector corresponding to ρ . By (1), all vertices of transmission $n-1$ have the same values given by X , say X_1 ; all vertices of transmission $2n-5$ have the same values given by X , say X_2 . Denote by X_3 the values of the vertices of transmission $n+1$ given by X . Assume $\tilde{X} = (X_1, X_2, X_3)^T$. Hence, by (1), we have

$$\begin{aligned} (\rho - (2 + 2 + (n-5)))X_1 &= 2X_1 + 2X_2 + (n-5)X_3, \\ (\rho - (3 + 2 + 2 \cdot (n-5)))X_2 &= 3X_1 + 2X_2 + 2 \cdot (n-5)X_3, \\ (\rho - (3 + 2 \cdot 2 + (n-6)))X_3 &= 3X_1 + 2 \cdot 2X_2 + (n-6)X_3. \end{aligned}$$

Transform the above equations into a matrix equation $(\rho I - A)\tilde{X} = 0$, we get

$$A = \begin{pmatrix} n+1 & 2 & n-5 \\ 3 & 2n-3 & 2n-10 \\ 3 & 4 & 2n-5 \end{pmatrix}.$$

Thus, $\rho_D(G)$ is the largest root of the following equation

$$\rho^3 - (5n-7)\rho^2 + (8n^2 - 31n + 56)\rho - 4n^3 + 26n^2 - 82n + 80 = 0.$$

Let $f(x) = x^3 - (5n-7)x^2 + (8n^2 - 31n + 56)x - 4n^3 + 26n^2 - 82n + 80$, then $f'(x) = 3x^2 - 2(5n-7)x + 8n^2 - 31n + 56$. Let $f'(x) = 0$, we get two roots x_1 and x_2 , such that $f'(x_1) = f'(x_2) = 0$, where

$$x_1 = \frac{5n-7 - \sqrt{n^2+23n-119}}{3}, x_2 = \frac{5n-7 + \sqrt{n^2+23n-119}}{3}.$$

Table 2: Some data of some graphs in \mathbb{NP}_2 .

G	$\rho_D(G)$	n	$\frac{2n^2+6n-28}{n}$
$K_5 \vee 6K_1$	27.2621	11	25.455
$K_3 \vee (K_2 + 3K_1)$	19.6847	8	18.5
$5K_1 \vee K_4$	21.8443	9	20.889
$K_3 \vee (K_{1,4} + K_1)$	22.0660	9	20.889
$K_3 \vee (K_{1,3} + K_2)$	22.0083	9	20.889
$K_2 \vee K_{2,5}$	22.0120	9	20.889
$K_3 \vee 4K_1$	16.4244	7	16
$K_2 \vee (K_{1,3} + K_1)$	16.9667	7	16
$K_2 \vee (K_{1,2} + K_2)$	16.6974	7	16
$K_1 \vee K_{2,4}$	16.6569	7	16

Consider $f(\frac{2n^2+6n-36}{n}) = -\frac{8(n^5-6n^4-70n^3+954n^2-4050n+5832)}{n^3} < 0$ for $n \geq 5$ and $\frac{2n^2+6n-36}{n} > x_2$ for $m \geq 5$, which implies $\rho_D(G) > \frac{2n^2+6n-36}{n}$, a contradiction.

The proof is complete. \blacksquare

Let $\mathbb{NP}_2 = \{K_2 \vee (K_{n-4} + 2K_1), K_5 \vee 6K_1, K_3 \vee (K_2 + 3K_1), 5K_1 \vee K_4, K_3 \vee (K_{1,4} + K_1), K_3 \vee (K_{1,3} + K_2), K_2 \vee K_{2,5}, K_3 \vee 4K_1, K_2 \vee (K_1 + K_{1,3}), K_2 \vee (K_{1,2} + K_2), K_1 \vee K_{2,4}\}$.

Lemma 2.7. ([15]). *Let G be a connected graph on $n \geq 4$ vertices and m edges with minimum degree $\delta \geq 2$. If*

$$m \geq \binom{n-2}{2} + 4,$$

then G is traceable from every vertex unless $G \in \mathbb{NP}_2$.

Theorem 2.8. *Let G be a connected graph on $n \geq 4$ vertices with minimum degree $\delta \geq 2$. If*

$$\rho_D(G) \leq \frac{2n^2 + 6n - 28}{n},$$

then G is traceable from every vertex.

Proof. From the proof of Theorem 2.6, we have $\rho_D(G) \geq 4(n-1) - \frac{4m}{n}$. By the condition of Theorem 2.8, we get

$$4(n-1) - \frac{4m}{n} \leq \rho_D(G) \leq \frac{2n^2 + 6n - 28}{n},$$

then $m \geq \binom{n-2}{2} + 4$. Suppose that G is not traceable from every vertex, by Lemma 2.7, we obtain $G \in \mathbb{NP}_2$. By direct calculation (see Table 2), we obtain that all graphs in \mathbb{NP}_2 except $G = K_2 \vee (K_{n-4} + 2K_1)$ satisfy $\rho_D(G) > \frac{2n^2+6n-28}{n}$, so we can get a contradiction.

For $G = K_2 \vee (K_{n-4} + 2K_1)$, let $X = (x_1, x_2, \dots, x_n)^T$ be the eigenvector corresponding to ρ . By (1), all vertices of transmission $n-1$ have the same values given by X , say X_1 ; all vertices of transmission $2n-4$ have the same values given by X , say X_2 . Denote by X_3 the values of the vertices of transmission $n+1$ given by X . Assume $\tilde{X} = (X_1, X_2, X_3)^T$. Hence, by (1), we have

$$(\rho - (1 + 2 + (n-4)))X_1 = X_1 + 2X_2 + (n-4)X_3,$$

$$(\rho - (2 + 2 + 2 \cdot (n-4)))X_2 = 2X_1 + 2X_2 + 2 \cdot (n-4)X_3,$$

$$(\rho - (2 + 2 \cdot 2 + (n - 5)))X_3 = 2X_1 + 2 \cdot 2X_2 + (n - 5)X_3.$$

Transform the above equations into a matrix equation $(\rho I - B)\tilde{X} = 0$, we get

$$B = \begin{pmatrix} n & 2 & n-4 \\ 2 & 2n-2 & 2n-8 \\ 2 & 4 & 2n-4 \end{pmatrix}.$$

Thus, $\rho_D(G)$ is the largest root of the following equation:

$$\rho^3 - (5n - 6)\rho^2 + (8n^2 - 28n + 44)\rho - 4n^3 + 24n^2 - 68n + 64 = 0.$$

Let $g(x) = x^3 - (5n - 6)x^2 + (8n^2 - 28n + 44)x - 4n^3 + 24n^2 - 68n + 64 = 0$, then $g'(x) = 3x^2 - 2(5n - 6)x + 8n^2 - 28n + 44$. Let $g'(x) = 0$, we have two values x_1 and x_2 , such that $g'(x_1) = g'(x_2) = 0$, where

$$x_1 = \frac{5n - 6 - \sqrt{n^2 + 24n - 96}}{3}, x_2 = \frac{5n - 6 + \sqrt{n^2 + 24n - 96}}{3}.$$

Consider $g(\frac{2n^2+6n-28}{n}) = -\frac{8(n^5-4n^4-67n^3+686n^2-2352n+2744)}{n^3} < 0$ for $n \geq 4$ and $\frac{2n^2+6n-28}{n} > x_2$ for $n \geq 4$, which implies $\rho_D(G) > \frac{2n^2+6n-28}{n}$, we can get a contradiction.

The proof is complete. ■

Lemma 2.9. ([4]). *Let G be a graph on n vertices and m edges. If*

$$m \geq \binom{n-1}{2},$$

then G contains a Hamilton path unless $G = K_{n-1} + v$. If the inequality is strict, then G contains a Hamilton cycle unless $G = K_{n-1} + e$.

Theorem 2.10. *Let G be a graph on n vertices, m edges and $\rho_D(G^C)$ be the distance signless Laplacian spectral radius of its complement. If*

$$\rho_D(G^C) \leq \frac{3n^2 - n + 10m - 2}{2n},$$

then G contains a Hamilton path unless $G = K_{n-1} + v$. If the inequality is strict, then G contains a Hamilton cycle unless $G = K_{n-1} + e$.

Proof. Because $\sigma(G^C) = \frac{1}{2} \sum_{v \in V(G)} \text{Tr}_{G^C}(v) \geq \frac{1}{2} \sum_{v \in V(G^C)} [(n-d(v)) + 2d(v)] = \frac{1}{2} \sum_{v \in V(G^C)} [n + d(v)] = \frac{1}{2}n(n-1) + m$. By Lemma 2.4, we have

$$\rho_D(G^C) \geq \frac{4\sigma(G^C)}{n} \geq 2(n-1) + \frac{4m}{n}.$$

Combining with the condition of Theorem 2.10, we get

$$2(n-1) + \frac{4m}{n} \leq \rho_D(G^C) \leq \frac{3n^2 - n + 10m - 2}{2n},$$

so $m \geq \binom{n-1}{2}$. Then by Lemma 2.9, we can obtain the conclusion.

The proof is complete. ■

References

- [1] S. Butler, F. Chung, Small spectral gap in the combinatorial Laplacian implies Hamiltonian, *Ann. Comb.* **13** (2010), 403-412.
- [2] V. Chvátal, On Hamilton's ideals, *J. Comb. Theorem B* **12** (1972), 163-168.
- [3] M. Ferrara, M. Jacobson, J. Powell, Characterizing degree-sum maximal nonhamiltonian bipartite graphs, *Discrete Math.* **26** (2012), 1088-1103.
- [4] M. Fiedler, V. Nikiforov, Spectral radius and Hamiltonicity of graphs, *Linear Algebra Appl.* **432** (2010), 2170-2173.
- [5] R.J. Gould, Advances on the Hamiltonian Problem-A survey, *Graphs Combin.* **19** (2003), 7-52.
- [6] R.J. Gould, Recent Advances on the Hamiltonian Problem: Survey III, *Graphs Combin.* **30** (2014), 1-46.
- [7] R.M. Karp, Reducibility among combinatorial problems, *in: Complexity of Computer Computations (New York, Plenum Press, 1972)*, 85-103.
- [8] M. Krivelevich, B. Sudakov, Sparse pseudo-random graphs are Hamiltonian, *J. Graph Theory* **42** (2003), 17-33.
- [9] R.F. Liu, W.C. Shiu, J. Xue, Sufficient spectral conditions on Hamiltonian and traceable graphs, *Linear Algebra Appl.* **467** (2015), 254-266.
- [10] J. Moon, L. Moser, On Hamiltonian bipartite graphs, *Israel J. Math.* **1** (1963), 163-165.
- [11] B. Ning, J. Ge, Spectral radius and Hamiltonian properties of graphs. *Linear and Multilinear Algebra* **63** (2015), 1520-1530.
- [12] R.D. Xing, B. Zhou, J.P. Li, On the distance signless Laplacian spectral radius of graphs, *Linear and Multilinear Algebra* **62** (2014), no.10, 1377-1387.
- [13] G.D. Yu, Y.Z. Fan, Spectral conditions for a graph to be Hamilton-connected, *Applied Mechanics and Materials* **336-338** (2013), 2329-2334.
- [14] B. Zhou, Signless Laplacian spectral radius and Hamiltonicity, *Linear Algebra Appl.* **432** (2010), 566-570.
- [15] Q.N. Zhou, L.G. Wang, Some sufficient spectral conditions on Hamilton-connected and traceable graphs, *Linear and Multilinear Algebra* <http://dx.doi.org/10.1080/03081087.2016.1182463>, 1-11.